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Two electrons in a homogeneous magnetic field: particular analytical solutions

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Abstract. Particular analytical solutions of the two-dimensional Schrödinger equation are described for two electrons (interacting with Coulomb potentials) in a homogeneous magnetic field B and an external oscillator potential with frequency ω_0. These exact solutions occur at an infinite and countable set of values of the quantity $\tilde{\omega} = \sqrt{\omega_0^2 + \frac{1}{4}(B/c)^2}$. Additionally, approximate closed-form solutions for the limits of small $\tilde{\omega}$ (perturbation theory in the electron–electron interaction) and large $\tilde{\omega}$ (harmonic approximation) are discussed and compared with the exact solutions.

1. Introduction

The solution of the two-dimensional Schrödinger equation for *one* electron in a homogeneous magnetic field B has been known already since the 20's [1]. An additional external oscillator potential does not produce significant complications. It is shown here, that for *two* electrons (interacting via Coulomb interaction) there are analytical solutions as well, provided the effective oscillator frequency $\tilde{\omega} = \sqrt{\omega_0^2 + \omega_L^2}$ belongs to a certain denumerably infinite set of values. Here ω_0 is the frequency of an admissible external oscillator potential, $\omega_L = \frac{1}{2}\omega_c = \frac{1}{2}B/c$ is the Larmor frequency, ω_c the cyclotron frequency, and c is the velocity of light§. This means that for a given ω_0 there are analytical solutions for a certain set of magnetic fields and vice versa. The sequence of admissible $\tilde{\omega}$ starts with a finite value of order 1 (depending on the angular momentum and the degree of excitation) and converges to zero. Thus the range of small ω_0 and B is covered particularly densely with solutions. Our method applies to singlet and triplet as well as to ground and excited states. An application of the underlying basic idea applied to the three-dimensional case without magnetic field is given in [2] and a numerical solution of the present problem is described in [3].

Possible applications of our solutions comprise the two-electron quantum dot and pairing problems of the two-dimensional electron gas in a magnetic field. It should also be useful for checking and assessing numerical and approximate methods for the two-dimensional electron gas with Coulomb correlations in a magnetic field.

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§ The cgs-system and atomic units $\hbar = m = e = 1$ are used.

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2. Exact solution

2.1. Decoupling

For the sake of a self-contained description, the decoupling of the Schrödinger equation into five easily solvable and one remaining equation of the type of a radial Schrödinger equation will be briefly reviewed (see also [3]). (It should be mentioned that, unless otherwise stated, this procedure applies to any gauge and dimension.) The Hamiltonian for the system in question reads

$$H = \sum_{i=1}^{2}\left\{\frac{1}{2}\left(p_i + \frac{1}{c}A(r_i)\right)^2 + \frac{1}{2}\omega_0^2 r_i^2\right\} + \frac{1}{|r_2 - r_1|} + H_{\mathrm{spin}} \tag{1}$$

where $H_{\mathrm{spin}} = g(s_1 + s_2)\cdot B$. Now we introduce relative and centre-of-mass coordinates

$$r = r_2 - r_1 \qquad R = \tfrac{1}{2}(r_1 + r_2) \tag{2}$$

which give rise to the definition of new momentum operators

$$p = \frac{1}{\mathrm{i}}\nabla_r = \tfrac{1}{2}(p_2 - p_1) \qquad P = \frac{1}{\mathrm{i}}\nabla_R = p_1 + p_2 . \tag{3}$$

If B is constant, A must be a linear function and we have

$$A(r) = A(r_2) - A(r_1) \qquad A(R) = \tfrac{1}{2}[A(r_1) + A(r_2)] . \tag{4}$$

In these coordinates, (1) reads

$$\begin{aligned} H &= 2\left\{\frac{1}{2}\left[p + \frac{1}{c}A_r\right]^2 + \frac{1}{2}\omega_r^2 r^2 + \frac{1}{2r}\right\} + \frac{1}{2}\left\{\frac{1}{2}\left[P + \frac{1}{c}A_R\right]^2 + \frac{1}{2}\omega_R^2 R^2\right\} + H_{\mathrm{spin}} \\ &\equiv 2H_r + \tfrac{1}{2}H_R + H_{\mathrm{spin}} \end{aligned} \tag{5}$$

where, for convenience, new parameters are defined as follows: $\omega_R = 2\omega_0, \omega_r = \frac{1}{2}\omega_0$, $A_R = 2A(R)$, and $A_r = \frac{1}{2}A(r)$. The special form of (5) allows a product ansatz for the eigenfunction

$$\Psi = \varphi(r)\cdot\xi(R)\cdot\chi(s_1, s_2) \tag{6}$$

and the eigenvalues have the form

$$E = 2\varepsilon_r + \tfrac{1}{2}\eta_R + E_{\mathrm{spin}} \tag{7}$$

where ε_r and η_R are the eigenvalues of the operators H_r and H_R, respectively. The *Pauli principle* demands that if $\varphi(r)$ is symmetric (antisymmetric) with respect to inversion $r \to -r$, then χ must be the singlet (triplet) spin state. No restrictions on $\xi(R)$ are imposed.

The eigensolutions of H_R are identical with those of a one-particle problem with modified parameters. For the sake of completeness and further reference, they will be given here (for derivation see [1]). From now on we restrict ourselves to the two-dimensional case, the gauge $A = \frac{1}{2}(B \times r)$ with B perpendicular to the plane, and polar coordinates (r, α):

$$\xi_{MN}(R) \propto r^{|M|}\,\mathrm{e}^{\mathrm{i}M\alpha} L_N^{|M|}(\tilde{\omega}_R R^2)\,\mathrm{e}^{-\frac{1}{2}\tilde{\omega}_R R^2} \tag{8}$$

$$\eta_{MN} = (2N + 1 + |M|)\tilde{\omega}_R + M\frac{B}{c} \tag{9}$$

where $L_N^{|M|}$ are associated Laguerre polynomials, $N = 0, 1, 2, \ldots, M = 0, \pm 1, \pm 2, \ldots$ is the angular momentum quantum number, and $\tilde{\omega}_R = 2\tilde{\omega}$.

2.2. Internal motion

The Schrödinger equation $H_r\varphi(\mathbf{r}) = \varepsilon_r\varphi(\mathbf{r})$ reads under the conditions specified above:

$$\left\{-\frac{1}{2}\left[r^{-1/2}\frac{\partial^2}{\partial r^2}r^{1/2} + \frac{1}{r^2}\left(\frac{\partial^2}{\partial\alpha^2} + \frac{1}{4}\right)\right] - \mathrm{i}\omega_{\mathrm{L}}\frac{\partial}{\partial\alpha} + \frac{1}{2}\left[\omega_r^2 + \frac{1}{4}\omega_{\mathrm{L}}^2\right]r^2 + \frac{1}{2r}\right\}\varphi = \varepsilon_r\varphi \tag{10}$$

which justifies the ansatz

$$\varphi = \frac{\mathrm{e}^{\mathrm{i}m\alpha}}{\sqrt{2\pi}}\frac{u(r)}{r^{1/2}} \qquad m = 0, \pm1, \pm2, \ldots. \tag{11}$$

Here $u(r)$ must satisfy the radial Schrödinger equation

$$\left\{-\frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}r^2} + \frac{1}{2}\left(m^2 - \frac{1}{4}\right)\frac{1}{r^2} + \frac{1}{2}\tilde{\omega}_r^2 r^2 + \frac{1}{2r}\right\}u(r) = \left[\varepsilon_r - \frac{1}{2}m\omega_{\mathrm{L}}\right]u(r) \tag{12}$$

where $\tilde{\omega}_r = \frac{1}{2}\tilde{\omega}$ and the solution is subject to the normalization condition $\int_0^\infty \mathrm{d}r|u(r)|^2 = 1$. Equation (12) is similar to that occurring in the three-dimensional problem without magnetic field (see equation (9) in [2]). We can therefore apply the same method, which will be summarized shortly. Substitute $\rho = \sqrt{\tilde{\omega}_r}r$ as well as

$$\varepsilon'' = \frac{2}{\tilde{\omega}_r}\left[\varepsilon_r - \tfrac{1}{2}m\omega_{\mathrm{L}}\right] \tag{13a}$$

$$u(\rho) = \mathrm{e}^{-\frac{1}{2}\rho^2}t(\rho) \tag{13b}$$

$$t(\rho) = \rho^{|m|+\frac{1}{2}}\sum_{\nu=0}^{\infty}a_\nu\rho^\nu \tag{13c}$$

into (12) and obtain the following recurrence relation for the coefficients a_ν:

$$a_0 \neq 0 \qquad a_1 = \frac{1}{2(|m| + \frac{1}{2})}\frac{1}{\sqrt{\tilde{\omega}_r}}a_0$$

and for $\nu \geqslant 2$

$$a_\nu = \frac{1}{\nu(\nu + 2|m|)}\left\{\frac{1}{\sqrt{\tilde{\omega}_r}}a_{\nu-1} + [2(\nu + |m| - 1) - \varepsilon'']a_{\nu-2}\right\}. \tag{14}$$

This relation allows us to express any coefficient in the form

$$a_\nu = F(|m|, \nu, \varepsilon'', \tilde{\omega}_r)a_0\,. \tag{15}$$

Normalizability of the wavefunction can be reached by termination of the power series at $\nu = n$. The conditions $a_n = 0$ *and* $a_{n+1} = 0$ are fulfilled if

$$F(|m|, n, \varepsilon'', \tilde{\omega}_r) = 0 \tag{16}$$

and

$$\varepsilon'' = 2(|m| + n)\,. \tag{17}$$

Both equations determine the spectrum of the allowed $\tilde{\omega}_r$ and ε_r. We proceed as follows. Once $F(|m|, n, \varepsilon'', \tilde{\omega}_r)$ is calculated for a particular n, we insert (17) into (16) and have an equation which determines $\tilde{\omega}_r$. For those $\tilde{\omega}_r$ we obtain the energies from (13a) and (17)

$$\varepsilon_r = \tfrac{1}{2}\tilde{\omega}_r 2\varepsilon'' + \tfrac{1}{2}m\omega_{\mathrm{L}} = (|m| + n)\tilde{\omega}_r + \tfrac{1}{2}m\omega_{\mathrm{L}}\,. \tag{18}$$

It is important that the solutions found in this way are not necessarily ground states. In the case that (16) and (17) have more than one solution for $\tilde{\omega}_r$, the solution with the smaller $\tilde{\omega}_r$ has zero nodes (ground state), that with the second largest $\tilde{\omega}_r$ has one node (first excited state), etc.

2.3. Results

The simplest solutions (first-order polynomials) are generated by $n = 2$. The corresponding F reads as

$$F(|m|, n = 2, \varepsilon'', \tilde{\omega}_r) = \frac{1}{4(|m|+1)}\left[\frac{1}{(2|m|+1)}\frac{1}{\tilde{\omega}} + 2(|m|+1) - \varepsilon''\right] \tag{19a}$$

and (16) and (17) give the spectrum

$$\tilde{\omega}_r = \frac{1}{2(2|m|+1)} \tag{19b}$$

$$\varepsilon_r = \frac{|m|+2}{2(2|m|+1)} + \tfrac{1}{2}m\omega_{\mathrm{L}}\,. \tag{19c}$$

Inserting this into (13) provides us with the (unnormalized) radial wavefunction

$$u(r) \propto \mathrm{e}^{-r^2/4(2|m|+1)} r^{(2|m|+1)/2}\left[1 + \frac{r}{(2|m|+1)}\right]. \tag{19d}$$

Analogously we obtain for $n = 3$.

$$F(|m|, n = 3, \varepsilon'', \tilde{\omega}_r) = \frac{1}{6(|m|+1)(2|m|+1)(2|m|+3)}\frac{1}{\sqrt{\tilde{\omega}}} *$$

$$\times\left[\frac{1}{2}\frac{1}{\tilde{\omega}_r} + 3(|m|+1)(2|m|+3) - \frac{1}{2}(6|m|+5)\varepsilon''\right] \tag{20a}$$

$$\tilde{\omega}_r = \frac{1}{4(4|m|+3)} \tag{20b}$$

$$\varepsilon_r = \frac{|m|+3}{4(4|m|+3)} + \tfrac{1}{2}m\omega_{\mathrm{L}} \tag{20c}$$

$$u(r) \propto \mathrm{e}^{-r^2/8(4|m|+3)} r^{(2|m|+1)/2}\left[1 + \frac{r}{(2|m|+1)} + \frac{r^2}{2(2|m|+1)(4|m|+3)}\right]. \tag{20d}$$

All states given above are ground states. A plot of some of these and some other eigenvalues is shown in figure 1 and ground-state solutions of the radial Schrödinger equation for some exactly soluble $\tilde{\omega}_r$ appear in figure 2. Observe that the radial function $u(r)$ contains all the information about the pair correlation function $G(r) = \langle\psi|\sum_{i<j}\delta(r_i - r_j - r)|\psi\rangle$ thanks to the relation

$$G(r) = |\varphi(r)|^2 = \frac{[u(r)]^2}{2\pi r}\,.$$

As follows from figure 2, both electrons prefer for small $\tilde{\omega}_r$ a certain distance, namely the classical distance r_0 defined below in (25). In figure 3 the electron density $n(r) = \langle\psi|\sum_i \delta(r - r_i)|\psi\rangle$ for some $\tilde{\omega}_r$ is shown. For large $\tilde{\omega}_r$ (what is equivalent to saying large maximum electron density) the electrons are located mainly around the origin in qualitative agreement with the behaviour of one electron. For small $\tilde{\omega}_r$, however, they arrange themselves on a ring with diameter r_0. Because their mutual distance approaches also the value r_0 (see figure 2), in this way they are maximizing their distance on the ring occupying antipodal positions. Thus they exhibit strongly correlated behaviour as to be expected in this limit. (See also the discussion by Laughlin [4] in the limit of high magnetic fields.)

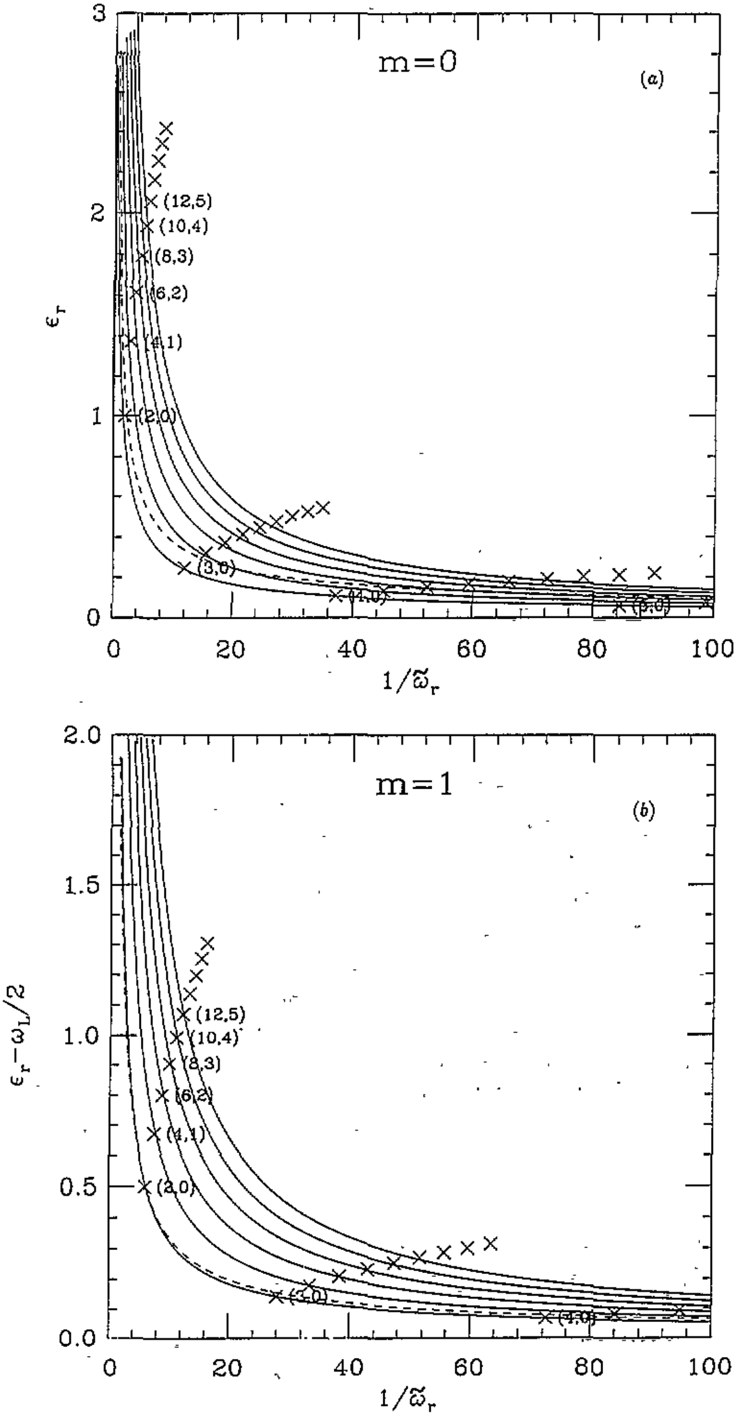

Figure 1. Some eigenvalues $(\varepsilon_r - \frac{1}{2}m\omega_L)$ for (*a*) $m = 0$ and (*b*) $m = 1$ versus $1/\tilde{\omega}_r$. Crosses are exact solutions and full curves are approximate solutions for small $\tilde{\omega}_r$ (28). The broken curve is the ground state in the limit of large $\tilde{\omega}_r$ (21). The numbers in parentheses to the right of the crosses are the termination index n and the number of zeros of the corresponding solution.

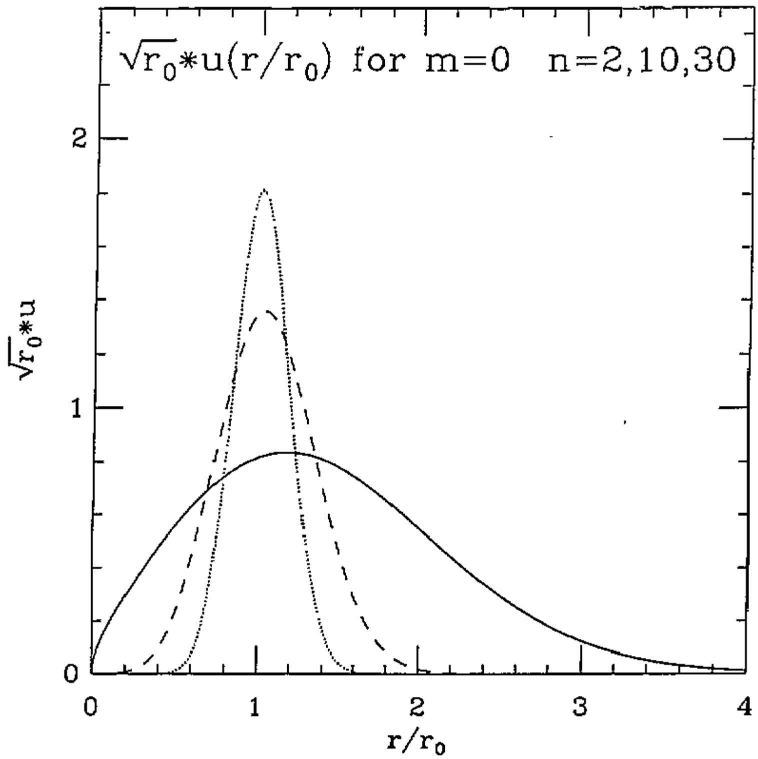

Figure 2. Radial part $u(r)$ of the ground state for $m = 0$ and $n = 2, 10, 30$, corresponding to $1/\tilde{\omega}_r = 2904.617$, and 29 312.4, respectively. r_0 is the classical electron–electron distance defined in (25).

3. Approximate solution for the relative motion

Despite the availability of particular exact solutions, the search for approximate solutions is justified for two reasons: firstly, they provide simple closed form solutions for *any* ω_0 and B, the applicability and accuracy of which can be checked by means of the exact solution given in section 3. Secondly, they can be applied to any electron number, where exact analytical solutions are not available and exact numerical solutions are practically not feasible.

3.1. High $\tilde{\omega}_r$

In this limit we can consider the electron–electron interaction in first-order perturbation theory. For the ground state of a given angular momentum m, one obtains (see [3])

$$\varepsilon_r = (|m| + 1)\tilde{\omega}_r + \tfrac{1}{2}m\omega_L + \frac{(2|m| - 1)!!}{(2|m|)!!}\tfrac{1}{2}\sqrt{\pi}\tilde{\omega}_r^{1/2}\,. \tag{21}$$

The first two terms are the energies of non-interacting electrons and the third term originates in the electron–electron interaction. As seen in figure 1, this approximation fails for small $\tilde{\omega}_r$, in particular for $m = 0$. In the range of intermediate $\tilde{\omega}_r$ ($\tilde{\omega}_r \sim 1$) it has the same accuracy as the result in the other limit (see section 3.2). Unfortunately, our method doesn't allow us to check (21) for large $\tilde{\omega}_r$, because it does not provide exact solutions in this limit.

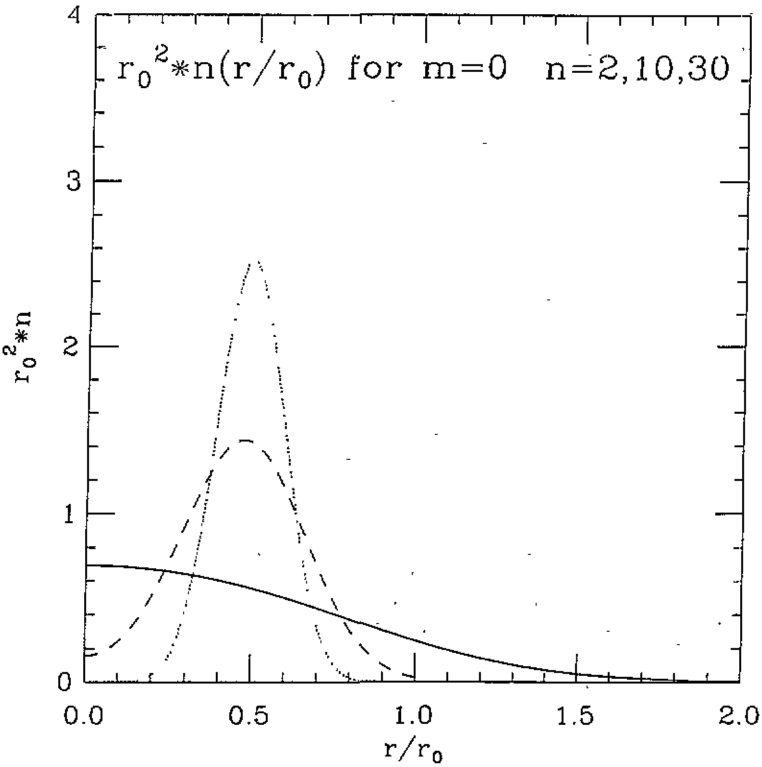

Figure 3. Electron density $n(r)$ corresponding to the solutions shown in figure 2.

3.2. Small $\tilde{\omega}_r$

The radial Schrödinger equation (12) can be written as

$$\left[-\frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}r^2} + V_{\mathrm{eff}}(r)\right]u(r) = \left[\varepsilon_r - \tfrac{1}{2}m\omega_{\mathrm{L}}\right]u(r) \tag{22}$$

with the effective potential

$$V_{\mathrm{eff}}(r) = \tfrac{1}{2}\tilde{\omega}_r^2 r^2 + \frac{1}{2}\left(m^2 - \frac{1}{4}\right)\frac{1}{r^2} + \frac{1}{2r}\,. \tag{23}$$

For small $\tilde{\omega}_r$, (22) can be solved approximately by expanding $V_{\mathrm{eff}}(r)$ around its (local) minimum r_{m}, determined by $\mathrm{d}/\mathrm{d}r\, V_{\mathrm{eff}}(r)|_{r=r_{\mathrm{m}}} = 0$, giving rise to the equation

$$r_{\mathrm{m}}^4 - \frac{1}{2\tilde{\omega}_r^2}r_{\mathrm{m}} - \frac{1}{\tilde{\omega}_r^2}\left(m^2 - \tfrac{1}{4}\right) = 0\,. \tag{24}$$

For small $\tilde{\omega}_r$, the third term of (24) can be neglected and the solution for (24) is then

$$r_0 = (2\tilde{\omega}_r^2)^{-1/3} = \left(\tfrac{1}{2}\tilde{\omega}^2\right)^{-1/3}. \tag{25}$$

Physically speaking, r_0 is the distance of two electrons in the ground state of our system in the classical limit. Obviously, $r_0 \to \infty$ as $\tilde{\omega}_r \to 0$, and r_0^{-1} can be treated as a small

Table 1. All solutions for the effective oscillator frequencies $\tilde{\omega}_r$ and the corresponding eigenvalues $(\varepsilon_r - \frac{1}{2}m\omega_L)$ for $n = 2$–20 for $m = 0$. N_r is the number of nodes of $u(r)$.

n	$1/\tilde{\omega}_r$	ε_r	N_r
2	0.200 000E+01	0.100 000E+01	0
3	0.120 000E+02	0.250 000E+00	0
4	0.370 880E+02	0.107 852E+00	0
	0.291 199E+01	0.137 363E+01	1
5	0.844 674E+02	0.591 944E−01	0
	0.155 326E+02	0.321 903E+00	1
6	0.161 253E+03	0.372 085E−01	0
	0.450 281E+02	0.133 250E+00	1
	0.371 853E+01	0.161 354E+01	2
7	0.274 552E+03	0.254 961E−01	0
	0.987 004E+02	0.709 217E−01	1
	0.187 477E+02	0.373 379E+00	2
8	0.431 472E+03	0.185 412E−01	0
	0.183 686E+03	0.435 527E−01	1
	0.523 811E+02	0.152 727E+00	2
	0.446 155E+01	0.179 310E+01	3
9	0.639 123E+03	0.140 818E−01	0
	0.307 090E+03	0.293 074E−01	1
	0.112 038E+03	0.803 299E−01	2
	0.217 493E+02	0.413 807E+00	3
	0.174 921E+04	0.800 359E−02	1
	0.105 055E+04	0.133 263E−01	2
	0.559 693E+03	0.250 137E−01	3
	0.244 586E+03	0.572 396E−01	4
	0.722 529E+02	0.193 764E+00	5
	0.646 710E+01	0.216 480E+01	6
15	0.334 860E+04	0.447 948E−02	0
	0.225 244E+04	0.665 946E−02	1
	0.141 609E+04	0.105 926E−01	2
	0.808 051E+03	0.185 632E−01	3
	0.396 399E+03	0.378 406E−01	4
	0.148 483E+03	0.101 022E+00	5
	0.299 411E+02	0.500 983E+00	6

parameter. We find an improved solution of (24) by inserting the ansatz $r_m = r_0 + \delta r = r_0(1 + \delta r/r_0)$ and keeping linear terms in the correction $\delta r/r_0$. In this way we obtain

$$r_m = r_0 + \tfrac{2}{3}\left(m^2 - \tfrac{1}{4}\right) + O(r_0^{-1}) \,. \tag{26}$$

With this approximate r_m, the harmonic approximation for V_{eff} reads

$$V_{\text{eff}}(r) = V_m + \tfrac{1}{2}\tilde{\omega}_m^2 (r - r_m)^2 \tag{27}$$

where

$$V_m = \tfrac{3}{4} r_0^{-1}\left[1 + \tfrac{2}{3}\left(m^2 - \tfrac{1}{4}\right) r_0^{-1} + O(r_0^{-2})\right]$$
$$\tilde{\omega}_m = \sqrt{\tfrac{3}{2}}\, r_0^{-3/2}\left[1 + \tfrac{2}{3}\left(m^2 - \tfrac{1}{4}\right) r_0^{-1} + O(r_0^{-2})\right]$$

and the approximate eigenvalue spectrum of (22) is

$$\varepsilon_r = V_m + \tilde{\omega}_m\left(n_{\text{ex}} + \tfrac{1}{2}\right) + \tfrac{1}{2}m\omega_L = \left[\tfrac{3}{4}\sqrt[3]{2}\,\tilde{\omega}_r^{2/3} + \sqrt{3}\,\tilde{\omega}_r\left(n_{\text{ex}} + \tfrac{1}{2}\right)\right]\left[1 + \tfrac{2}{3}\sqrt[3]{2}\,\tilde{\omega}_r^{2/3}\left(m^2 - \tfrac{1}{4}\right)\right] + \tfrac{1}{2}m\omega_L \tag{28}$$

Table 2. As in the table 1, but for $m = 1$

n	$1/\tilde{\omega}_r$	$\varepsilon_r - \frac{1}{2}\omega_L$	N_r
2	0.600 000E+01	0.500 000E+00	0
3	0.280 000E+02	0.142 857E+00	0
4	0.725 576E+02	0.689 107E−01	0
	0.744 236E+01	0.671 830E+00	1
5	0.146 604E+03	0.409 266E−01	0
	0.333 961E+02	0.179 662E+00	1
6	0.257 194E+03	0.272 168E−01	0
	0.840 644E+02	0.832 695E−01	1
	0.874 155E+01	0.800 773E+00	2
7	0.411 420E+03	0.194 448E−01	0
	0.166 223E+03	0.481 280E−01	1
	0.383 564E+02	0.208 570E+00	2
8	0.616 386E+03	0.146 012E−01	0
	0.286 870E+03	0.313 730E−01	1
	0.947 990E+02	0.949 377E−01	2
	0.994 462E+01	0.905 012E+00	3
9	0.879 199E+03	0.113 740E−01	0
	0.453 076E+03	0.220 713E−01	1
	0.184 721E+03	0.541 356E−01	2
	0.430 035E+02	0.232 539E+00	3
10	0.120 697E+04	0.911 375E−02	0
	0.671 937E+03	0.163 706E−01	1
	0.315 069E+03	0.349 130E−01	2
	0.104 949E+03	0.104 813E+00	3
	0.110 772E+02	0.993 032E+00	4
11	0.160 680E+04	0.746 824E−02	0
	0.950 554E+03	0.126 242E−01	1
	0.492 895E+03	0.243 460E−01	2
	0.202 336E+03	0.593 073E−01	3
	0.474 112E+02	0.253 105E+00	4
12	0.208 582E+04	0.623 257E−02	0
	0.129 604E+04	0.100 306E−01	1
	0.725 285E+03	0.179 240E−01	2
	0.342 070E+03	0.380 039E−01	3
	0.114 636E+03	0.113 402E+00	4
	0.121 551E+02	0.106 951E+01	5
13	0.265 112E+04	0.528 080E−02	0
	0.171 549E+04	0.816 091E−02	1
	0.101 934E+04	0.137 343E−01	2
	0.531 188E+03	0.263 560E−01	3
	0.219 232E+03	0.638 592E−01	4
	0.516 277E+02	0.271 172E+00	5
14	0.330 981E+04	0.453 198E−02	0
	0.221 603E+04	0.676 885E−02	1
	0.138 217E+04	0.108 525E−01	2
	0.776 774E+03	0.193 106E−01	3
	0.368 074E+03	0.407 527E−01	4
	0.123 945E+03	0.121 022E+00	5
	0.131 888E+02	0.113 733E+01	6
15	0.406 902E+04	0.393 215E−02	0
	0.280 477E+04	0.570 457E−02	1
	0.182 088E+04	0.878 696E−02	2
	0.108 593E+04	0.147 340E−01	3
	0.568 190E+03	0.281 596E−01	4
	0.235 528E+03	0.679 325E−01	5
	0.556 868E+02	0.287 321E+00	6

where $n_{ex} = 0, 1, 2, \ldots$ is the degree of excitation of the state.

So far the following problem has been disregarded completely: for $m = 0$ the effective potential has a negative pole at $r = 0$ (see figure 4). This endangers our harmonic approximation. It turns out, however, that despite the pole the curves retain a local minimum if $\tilde{\omega}_r < \frac{3}{4}\sqrt{3} = 1.299$, and that even for the worst case, for which an exact solution exists ($n = 2, \tilde{\omega}_r = 2, \varepsilon_r = 1$), the energy in the harmonic approximation $\varepsilon_r = 0.892\,264$ is not that bad. As an empirical result, it should be mentioned that better results for $m = 0$ are obtained by simply neglecting the centrifugal potential (second term in (23) and second bracket in (28)). Then we obtain $\varepsilon_r = 1.028\,29$, which agrees pretty well with the exact value $\varepsilon_r = 1$ in the worst case considered above. For $m > 0$, however, consideration of the centrifual term improves the agreement with the exact result considerably.

3.3. Interpolation

We are now looking for an interpolation formula which fulfills the high and small $\tilde{\omega}_r$-limits (21) and (28) and which gives acceptable accuracy for intermediate $\tilde{\omega}_r$. We propose that

$$f(\tilde{\omega}_r) = \frac{f_\infty(\tilde{\omega}_r)\tilde{\omega}_r + f_0(\tilde{\omega}_r)\tilde{\omega}_r^{-1}}{\tilde{\omega}_r + \tilde{\omega}_r^{-1}}$$

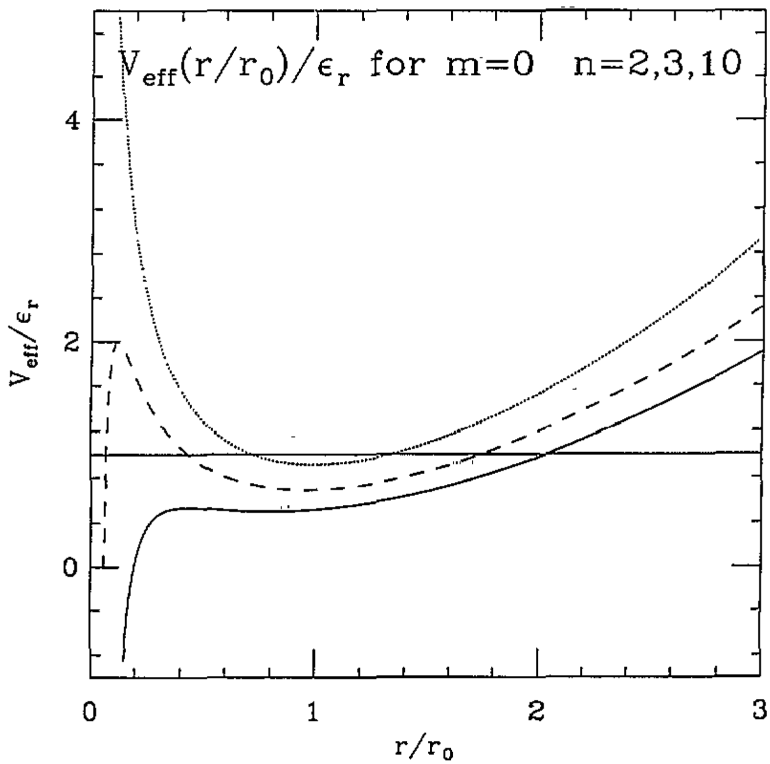

Figure 4. Effective potential V_{eff} divided by the ground-state energy ε_r for $m = 0$ for three different $\tilde{\omega}_r$ corresponding to the exact solutions for $n = 2$ (full line), 3 (broken line), and 10 (dotted line). The corresponding values for $1/\tilde{\omega}_r$ are 2,12, and 904.617. (The dotted curve also goes to $-\infty$ for $r \to 0$, but this branch of the curve is so close to the ordinate that it cannot be resolved from the axis.) The horizontal line is the eigenvalue ε_r for all three curves alike.

where $f(\tilde{\omega}_r) = \varepsilon_r - \frac{1}{2}m\omega_L$ and f_∞ and f_0 are the results given in (21) and (28), respectively. The maximum error of this formula for the available exact solutions is 6% for $m = 0$ and 3% for $m = 1$. This result is important as an estimate of errors for a forthcoming calculation of the N-electron quantum dot, where this interpolation will be used.

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